

Positivity for pre-canonical bases in type A

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1 Root systems and Weyl groups

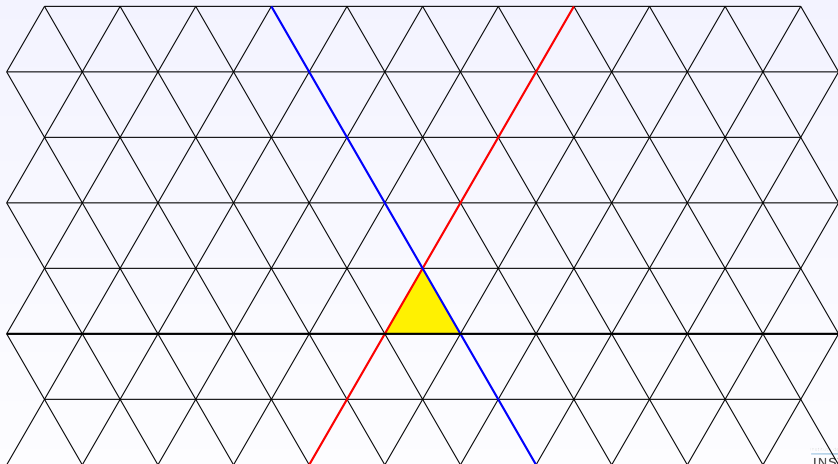
2 Hecke algebra and Kazhdan–Lusztig basis

3 Spherical Hecke algebra

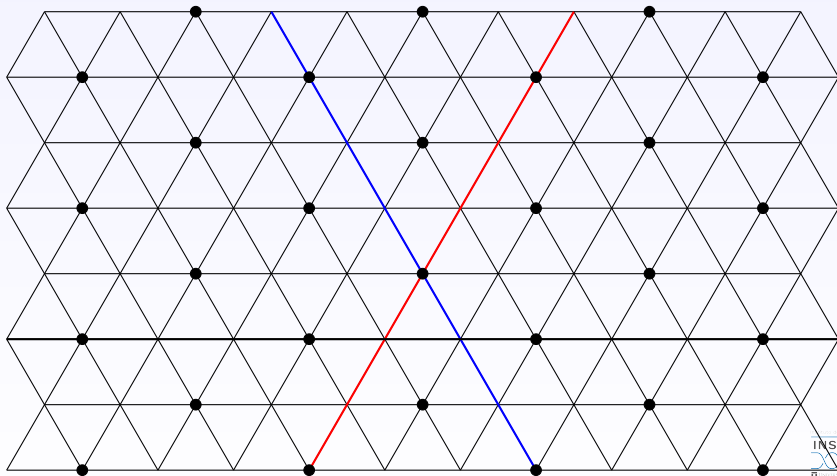
4 Pre-canonical bases

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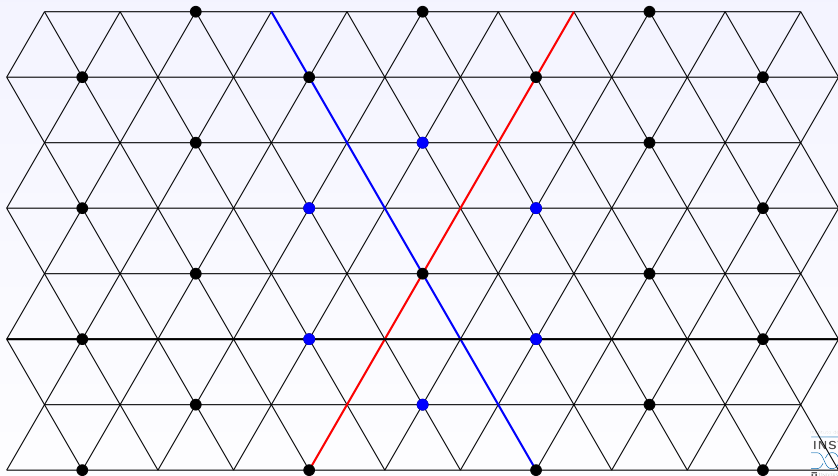
Affine Weyl group of type A_2



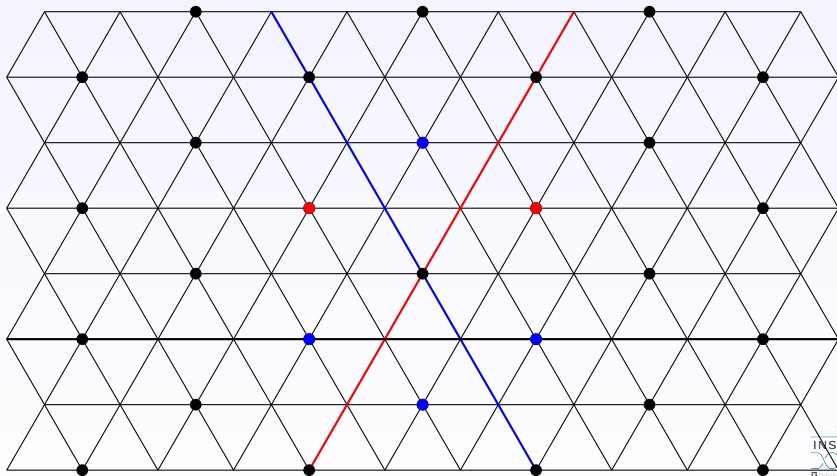
Root lattice $\mathbb{Z}\Phi$, roots Φ , simple roots Δ and positive and negative roots $\Phi^\pm$



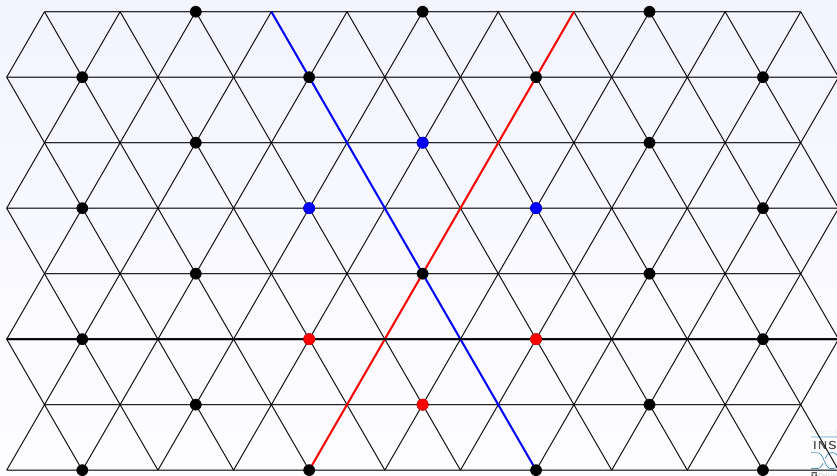
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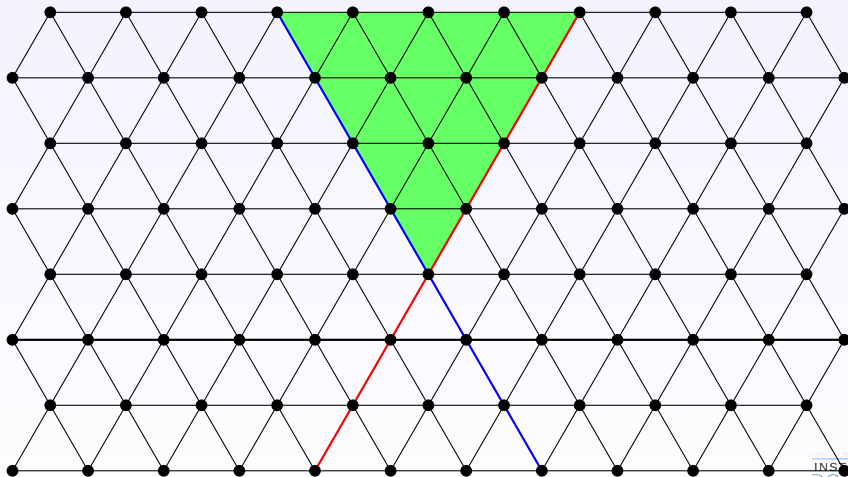
Root lattice $\mathbb{Z}\Phi$, roots Φ , simple roots Δ and positive and negative roots $\Phi^\pm$



Weight lattice X , fundamental weights ϖ_i , dominant weights X^+



Weight lattice X , fundamental weights ϖ_i , dominant weights X^+



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Hecke algebra

Definition 1

Let (W, S) be a Coxeter system. The Hecke algebra $\mathcal{H} = \mathcal{H}(W, S)$ is the $\mathbb{Z}[v, v^{-1}]$ -associative algebra generated by $\{\mathbf{H}_s \mid s \in S\}$ under the relations

$$\begin{aligned}(\mathbf{H}_s + v)(\mathbf{H}_s - v^{-1}) &= 0 && \text{for } s \in S, \\ \mathbf{H}_s \mathbf{H}_t \mathbf{H}_s \cdots &= \mathbf{H}_t \mathbf{H}_s \mathbf{H}_t \cdots && m_{s,t} \text{ times for } s, t \in S.\end{aligned}$$

For any $x \in W$ and $x = s_{i_1} s_{i_2} \cdots s_{i_r}$ a reduced expression for x , set

$$\mathbf{H}_x := \mathbf{H}_{s_{i_1}} \mathbf{H}_{s_{i_2}} \cdots \mathbf{H}_{s_{i_r}} \quad (1)$$

(this does not depend on the choice of reduced expression.)

Theorem 2.1

$\mathcal{H}$ is a free $\mathbb{Z}[v, v^{-1}]$ -module with basis $(\mathbf{H}_x)_{x \in W}$ (called the standard basis).

Kazhdan–Lusztig basis

By relation $\mathbf{H}_s^2 = (v^{-1} - v)\mathbf{H}_s + 1$ for $s \in S$, notice that

$$(\mathbf{H}_s + v - v^{-1})(\mathbf{H}_s) = \mathbf{H}_s^2 + (v - v^{-1})\mathbf{H}_s = (v^{-1} - v)\mathbf{H}_s + (v - v^{-1})\mathbf{H}_s + 1 = 1. \quad (2)$$

So, $\mathbf{H}_s$ are invertible for all $s \in S$ and therefore if $x = s_{i_1} s_{i_2} \cdots s_{i_r}$ is a reduced expression,

$$\mathbf{H}_x^{-1} = \mathbf{H}_{s_{i_r}}^{-1} \cdots \mathbf{H}_{s_{i_2}}^{-1} \mathbf{H}_{s_{i_1}}^{-1}$$

is invertible too!

Definition 2

The Kazhdan–Lusztig involution $\bar{\cdot} : \mathcal{H} \rightarrow \mathcal{H}$, $h \mapsto \bar{h}$, the $\mathbb{Z}$ -linear map defined as follows

$$\begin{aligned} \bar{v} &= v^{-1}; \\ \bar{\mathbf{H}}_s &= \mathbf{H}_s^{-1}, \end{aligned} \quad (3)$$

and extended to product as ring isomorphism.

Kazhdan–Lusztig basis

Theorem 2.2 (KL79)

There exist a unique basis $\{\underline{\mathbf{H}}_x | x \in W\}$ such that

- ① $\underline{\mathbf{H}}_x$ is self dual, i.e. $\overline{\underline{\mathbf{H}}_x} = \underline{\mathbf{H}}_x$.
- ② $\underline{\mathbf{H}}_x = \mathbf{H}_x + \sum_{y < x} h_{y,x} \mathbf{H}_y$, for some $h_{y,x} \in v\mathbb{Z}[v]$.

The coefficients $h_{y,x} \in v\mathbb{Z}[v]$ are called the Kazhdan–Lusztig polynomials.

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Spherical Hecke algebra

Let $\mathcal{H}$ be the Hecke algebra of W_a over $\mathbb{Z}[v, v^{-1}]$ with standard basis $\{\mathbf{H}_x\}$ and Kazhdan-Lusztig basis $\{\underline{\mathbf{H}}_x\}$. We define

$$\begin{aligned}\tau \mathcal{H} &:= \underline{\mathbf{H}}_{\tau(\omega_0)} \mathcal{H} \\ \mathcal{H}^\sigma &:= \mathcal{H} \underline{\mathbf{H}}_{\sigma(\omega_0)} \\ {}^\sigma \mathcal{H}^\tau &:= {}^\tau \mathcal{H} \cap \mathcal{H}^\sigma\end{aligned}$$

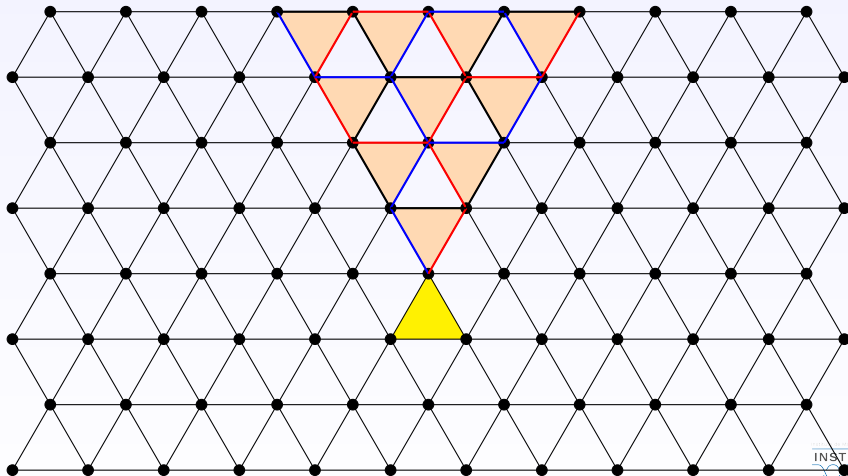
For a finite subgroup H of W_a we denote by $\pi_H(q)$ the Poincaré polynomial of H , defined as $\pi_H(q) = \sum_{w \in H} q^{\ell(w)}$.

$$\widetilde{\mathcal{H}} := \bigoplus_{\sigma \in \Omega} {}^1 \mathcal{H}^\sigma$$

where multiplication $\mathcal{H}^\tau \times \mathcal{H}^\sigma \rightarrow \mathcal{H}^{\tau\sigma}$ is defined via

$$\begin{aligned}\mathcal{H}^\tau \times \mathcal{H}^\sigma &\xrightarrow{\sim} \mathcal{H}^\tau \times {}^\tau \mathcal{H}^{\tau\sigma} \rightarrow \mathcal{H}^{\tau\sigma} \\ (x, y) &\rightarrow (x, \tau(y)) \rightarrow \frac{v^{\ell(w_0)}}{\pi_{W_f}(v^2)} x\tau(y)\end{aligned}$$

$\theta(\lambda)$ elements



Standard basis and Kazhdan–Lusztig basis for the spherical algebra

Standard basis

For $\lambda \in X^+$ we define

$$\mathbf{H}_\lambda := \sum_{x \in W^{\theta(\lambda)} W_\sigma} v^{\ell(\theta(\lambda)) - \ell(x)} \mathbf{H}_x.$$

The set $(\mathbf{H}_\lambda)_{\lambda \in X^+}$ is the standard basis of the spherical algebra.

Kazhdan–Lusztig basis for the spherical algebra

$$\underline{\mathbf{H}}_\lambda = \underline{\mathbf{H}}_{\theta(\lambda)}.$$

And as before we have Kazhdan–Lusztig polynomials:

$$\underline{\mathbf{H}}_\lambda = \mathbf{H}_\lambda + \sum_{\mu < \lambda} h_{\mu, \lambda} \mathbf{H}_\mu,$$

where $h_{\mu, \lambda} = h_{\theta(\mu), \theta(\lambda)}$.

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Pre-canonical bases

We can define an additional basis of $\tilde{\mathcal{H}}$ as follows

$$\mathbf{N}_\lambda := \sum_{\mu \leq \lambda} v^{\ell(\theta(\lambda)) - \ell(\theta(\mu))} \mathbf{H}_\mu$$

where the Kazhdan-Lusztig basis expansion is expressed as:

$$\underline{\mathbf{H}}_\lambda = \sum_{\mu \leq \lambda} a_{\lambda, \mu}(v) \mathbf{N}_\mu$$

The coefficients $a_{\lambda, \mu}(v) \in \mathbb{Z}[q]$ are called the Atomic polynomials. In general, we say that $\underline{\mathbf{H}}_\lambda$ admits an atomic decomposition if $a_{\lambda, \mu}(v) \in \mathbb{Z}_{\geq 0}$.

This definition serves as a motivation to define a new family of basis for the spherical Hecke algebra:

Definition

For $i \geq 2$ and $\lambda \in X^+$ we define

$$\mathbf{N}_\lambda^i := \sum_{I \subset \Phi \geq i} (-q)^{|I|} \tilde{\underline{\mathbf{H}}}_{\lambda - \sum_{\alpha \in I} \alpha}, \quad \text{and} \quad \mathbf{N}_\lambda^1 := \mathbf{H}_\lambda.$$

The sets $(\mathbf{N}_\lambda^i)_{\lambda \in X^+}$ are bases to the spherical Hecke algebra for each $i \geq 1$ and $q = v^2$.

Pre-canonical bases

Conjecture (Libedinsky, Patimo, Plaza)

For $i \geq 2$ and $\lambda \in X^+$

$$\mathbf{N}_\lambda^{i+1} = \sum_{\mu \in X^+} r_{\mu, \lambda} \mathbf{N}_\mu^i,$$

the polynomials $r_{\mu, \lambda} \in \mathbb{Z}_{\geq 0}[q]$.

M - elements

Definition

For $A \subset \Phi$ and $\lambda \in X$, we define

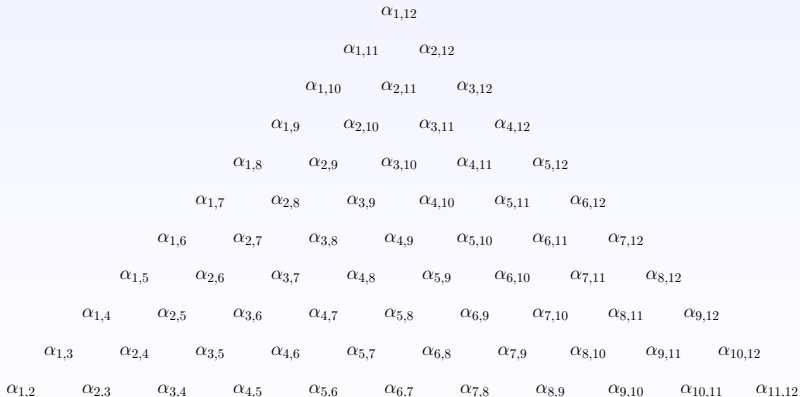
$$\mathbf{M}_{\lambda}^A := \sum_{I \subset A} (-q)^{|I|} \tilde{\mathbf{H}}_{\lambda - \sum_{\alpha \in I} \alpha}.$$

On $\mathbf{M}$ - elements we have the following relation

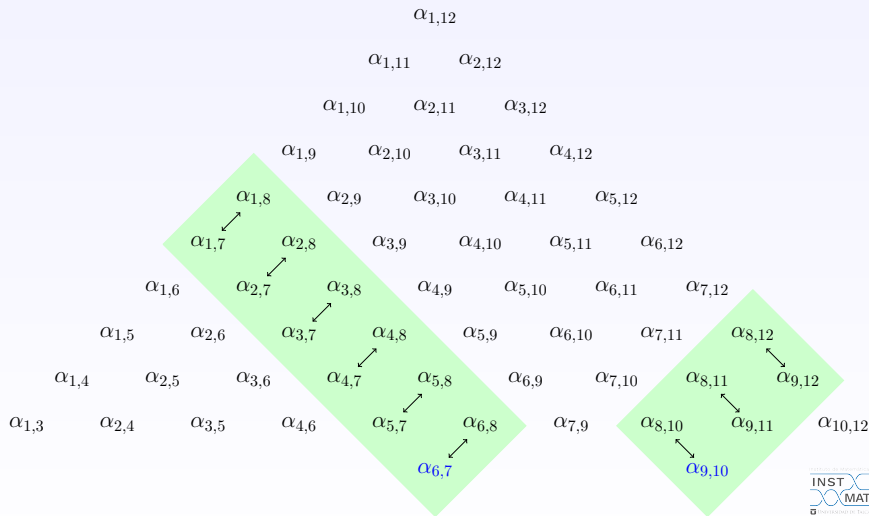
$$\mathbf{M}_{\lambda}^A = \mathbf{M}_{\lambda}^{A \setminus \{\alpha\}} - q \mathbf{M}_{\lambda - \alpha}^{A \setminus \{\alpha\}}.$$

In particular, if $A = \Phi^{\geq i}$, then $\mathbf{M}_{\lambda}^A = \mathbf{N}_{\lambda}^i$.

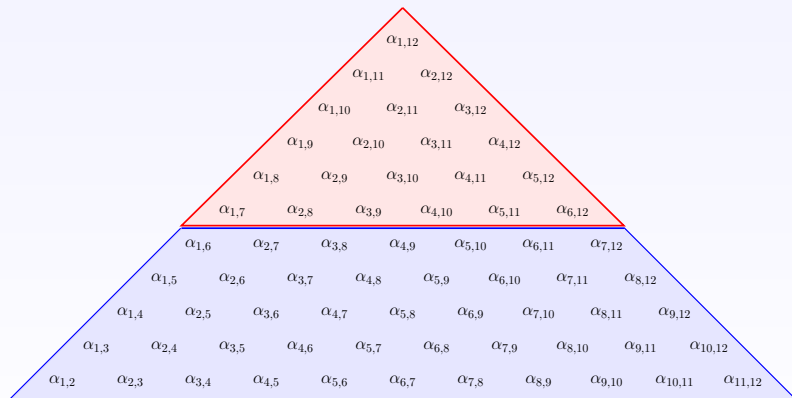
Positive roots in A_{12}



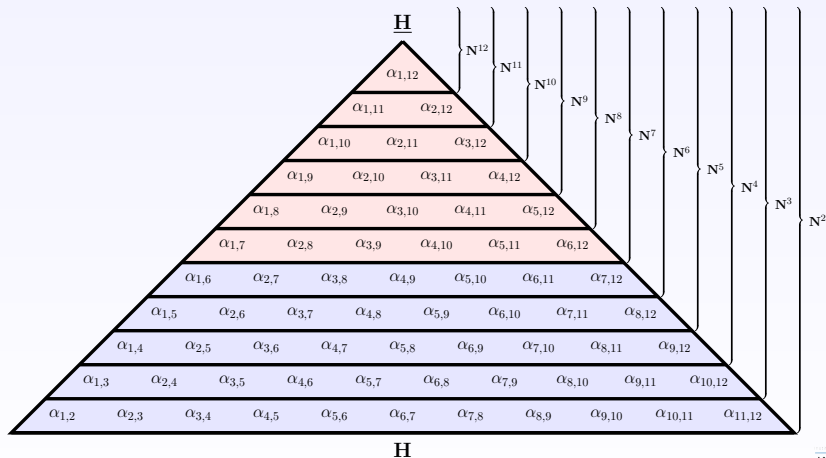
Ex. action of s_{α_8} on $\Phi^{\geq 3}$



Two subsets of positive roots



Pre-canonical bases



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Positivity of Pre-canonical bases

We will show the positivity of pre-canonical bases by using an example of the arguments that were used to prove the positivity conjecture.

Let $n = 14$, $\alpha_{i,j} = \alpha_{8,10}$ ($\text{ht}(\alpha_{8,10}) = 3$) and $\lambda = [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 1, 4]_{\varpi}$

We will describe the positive expansion of the element $M_{\lambda}^{\succ \alpha_{8,10}}$ in terms of the M -elements $M^{\succ \alpha}$ for $\alpha \in [\alpha_{1,3}, \alpha_{8,10}]$.

Reverse decomposition

$$\begin{aligned} \mathbf{M}_{\lambda}^{\succ \alpha_{8,10}} &= \mathbf{M}_{\lambda}^{\succ \alpha_{8,10}} [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\ &\quad - q \mathbf{M}_{\lambda - \alpha_{8,10}}^{\succ \alpha_{8,10}} [0, 3, 0, 2, 1, 2, 1, -1, 0, -1, 1, 0, 1, 4]_{\varpi} \end{aligned}$$

Reverse decomposition

$$\begin{aligned} \mathbf{M}_{\lambda}^{\gamma^{\alpha_{8,10}}} &= \mathbf{M}_{\lambda}^{\gamma^{\alpha_{8,10}}} [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\ &\quad - q^2 \mathbf{M}_{\lambda - \alpha_{5,10}}^{\gamma^{\alpha_{8,10}}} [0, 3, 0, 3, 0, 2, 0, 0, 0, -1, 1, 0, 1, 4]_{\varpi} \end{aligned}$$

Reverse decomposition

$$\begin{aligned}
 \mathbf{M}_{\lambda}^{\gamma^{\alpha_{8,10}}} &= \mathbf{M}_{\lambda}^{\gamma^{\alpha_{8,10}}} && [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\
 &-q^3 \mathbf{M}_{\lambda-\alpha_{5,10}-\alpha_{7,9}}^{\gamma^{\alpha_{8,10}}} && [0, 3, 0, 3, 0, 3, -1, 0, -1, 0, 1, 0, 1, 4]_{\varpi} \\
 &-q^3 \mathbf{M}_{\lambda-\alpha_{5,12}}^{\gamma^{\alpha_{8,10}}} && [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, -1, 2, 4]_{\varpi} \\
 &+q^4 \mathbf{M}_{\lambda-\alpha_{5,12}-\alpha_{7,9}}^{\gamma^{\alpha_{8,10}}} && [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, -1, 2, 4]_{\varpi}
 \end{aligned}$$

Reverse decomposition

$$\begin{aligned}
 \mathbf{M}_{\lambda}^{\succ \alpha_{8,10}} &= \mathbf{M}_{\lambda}^{\succ \alpha_{8,10}} && [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\
 &- q^3 \mathbf{M}_{\lambda - \alpha_{5,10} - \alpha_{7,9}}^{\succ \alpha_{8,10}} && [0, 3, 0, 3, 0, 3, -1, 0, -1, 0, 1, 0, 1, 4]_{\varpi} \\
 &- q^4 \mathbf{M}_{\lambda - \alpha_{5,14}}^{\succ \alpha_{8,10}} && [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{7,9}}^{\succ \alpha_{8,10}} && [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, 0, 1, 3]_{\varpi}
 \end{aligned}$$

Reverse decomposition

$$\begin{aligned}
 \mathbf{M}_{\lambda}^{\gamma_{\alpha_{8,10}}} &= \mathbf{M}_{\lambda}^{\gamma_{\alpha_{8,10}}} && [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\
 &- q^4 \mathbf{M}_{\lambda - \alpha_{5,10} - \alpha_{4,9}}^{\gamma_{\alpha_{8,10}}} && [0, 3, 1, 2, 0, 2, 0, 0, -1, 0, 1, 0, 1, 4]_{\varpi} \\
 &- q^4 \mathbf{M}_{\lambda - \alpha_{5,14}}^{\gamma_{\alpha_{8,10}}} && [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{7,9}}^{\gamma_{\alpha_{8,10}}} && [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, 0, 1, 3]_{\varpi}
 \end{aligned}$$

Reverse decomposition

$$\begin{aligned}
 \mathbf{M}_{\lambda}^{\gamma_{\alpha_{8,10}}} &= \mathbf{M}_{\lambda}^{\gamma_{\alpha_{8,10}}} & [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\
 &-q^4 \mathbf{M}_{\lambda-\alpha_{5,10}-\alpha_{4,9}-\alpha_{6,8}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 1, 2, 1, 1, 0, -1, 0, 0, 1, 0, 1, 4]_{\varpi} \\
 &-q^4 \mathbf{M}_{\lambda-\alpha_{5,14}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &+q^5 \mathbf{M}_{\lambda-\alpha_{5,14}-\alpha_{7,9}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, 0, 1, 3]_{\varpi} \\
 &-q^5 \mathbf{M}_{\lambda-\alpha_{5,10}-\alpha_{4,11}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 1, 2, 0, 2, 0, 0, 0, -1, 0, 1, 1, 4]_{\varpi} \\
 &+q^6 \mathbf{M}_{\lambda-\alpha_{5,10}-\alpha_{4,11}-\alpha_{6,8}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 1, 2, 1, 1, 0, -1, 1, -1, 0, 1, 1, 4]_{\varpi}
 \end{aligned}$$

Reverse decomposition

$$\begin{aligned}
 \mathbf{M}_{\lambda}^{\gamma_{\alpha_{8,10}}} &= \mathbf{M}_{\lambda}^{\gamma_{\alpha_{8,10}}} & [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\
 &- q^4 \mathbf{M}_{\lambda - \alpha_{5,10} - \alpha_{4,9} - \alpha_{6,8}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 1, 2, 1, 1, 0, -1, 0, 0, 1, 0, 1, 4]_{\varpi} \\
 &- q^4 \mathbf{M}_{\lambda - \alpha_{5,14}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{7,9}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, 0, 1, 3]_{\varpi} \\
 &- q^6 \mathbf{M}_{\lambda - \alpha_{5,12} - \alpha_{4,11}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 1, 2, 0, 2, 0, 0, 0, 0, -1, 0, 2, 4]_{\varpi} \\
 &+ q^7 \mathbf{M}_{\lambda - \alpha_{5,12} - \alpha_{4,11} - \alpha_{6,8}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 1, 2, 1, 1, 0, -1, 1, 0, -1, 0, 2, 4]_{\varpi}
 \end{aligned}$$

Reverse decomposition

$$\begin{aligned}
 \mathbf{M}_{\lambda}^{\gamma_{\alpha_{8,10}}} &= \mathbf{M}_{\lambda}^{\gamma_{\alpha_{8,10}}} & [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\
 &- q^4 \mathbf{M}_{\lambda - \alpha_{5,10} - \alpha_{4,9} - \alpha_{6,8}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 1, 2, 1, 1, 0, -1, 0, 0, 1, 0, 1, 4]_{\varpi} \\
 &- q^4 \mathbf{M}_{\lambda - \alpha_{5,14}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{7,9}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, 0, 1, 3]_{\varpi} \\
 &- q^6 \mathbf{M}_{\lambda - \alpha_{5,12} - \alpha_{4,13}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, -1, 1, 5]_{\varpi} \\
 &+ q^7 \mathbf{M}_{\lambda - \alpha_{5,12} - \alpha_{4,13} - \alpha_{6,8}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 1, 2, 1, 1, 0, -1, 1, 0, 0, -1, 1, 5]_{\varpi}
 \end{aligned}$$

Reverse decomposition

$$\begin{aligned}
 \mathbf{M}_{\lambda}^{\gamma_{\alpha_{8,10}}} &= \mathbf{M}_{\lambda}^{\gamma_{\alpha_{8,10}}} & [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\
 &- q^4 \mathbf{M}_{\lambda - \alpha_{5,10} - \alpha_{4,9} - \alpha_{6,8}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 1, 2, 1, 1, 0, -1, 0, 0, 1, 0, 1, 4]_{\varpi} \\
 &- q^4 \mathbf{M}_{\lambda - \alpha_{5,14}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{7,9}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, 0, 1, 3]_{\varpi} \\
 &- q^7 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 4]_{\varpi} \\
 &+ q^8 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13} - \alpha_{6,8}}^{\gamma_{\alpha_{8,10}}} & [0, 3, 1, 2, 1, 1, 0, -1, 1, 0, 0, 0, 0, 4]_{\varpi}
 \end{aligned}$$

Reverse decomposition

$$\begin{aligned}
 \mathbf{M}_{\lambda}^{\gamma \alpha_{8,10}} &= \mathbf{M}_{\lambda}^{\gamma \alpha_{8,10}} & [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\
 &- q^4 \mathbf{M}_{\lambda - \alpha_{5,10} - \alpha_{4,9} - \alpha_{6,8}}^{\gamma \alpha_{8,10}} & [0, 3, 1, 2, 1, 1, 0, -1, 0, 0, 1, 0, 1, 4]_{\varpi} \\
 &- q^4 \mathbf{M}_{\lambda - \alpha_{5,14}}^{\gamma \alpha_{8,10}} & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{7,9}}^{\gamma \alpha_{8,10}} & [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, 0, 1, 3]_{\varpi} \\
 &- q^7 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13}}^{\gamma \alpha_{8,10}} & [0, 3, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 4]_{\varpi} \\
 &+ q^9 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13} - \alpha_{6,8} - \alpha_{5,7}}^{\gamma \alpha_{8,10}} & [0, 3, 1, 3, 0, 1, -1, 0, 1, 0, 0, 0, 0, 4]_{\varpi}
 \end{aligned}$$

Reverse decomposition

$$\begin{aligned}
 \mathbf{M}_{\lambda}^{\gamma \alpha_{8,10}} &= \mathbf{M}_{\lambda}^{\gamma \alpha_{8,10}} & [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\
 &- q^4 \mathbf{M}_{\lambda - \alpha_{5,10} - \alpha_{4,9} - \alpha_{6,8}}^{\gamma \alpha_{8,10}} & [0, 3, 1, 2, 1, 1, 0, -1, 0, 0, 1, 0, 1, 4]_{\varpi} \\
 &- q^4 \mathbf{M}_{\lambda - \alpha_{5,14}}^{\gamma \alpha_{8,10}} & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{7,9}}^{\gamma \alpha_{8,10}} & [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, 0, 1, 3]_{\varpi} \\
 &- q^7 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13}}^{\gamma \alpha_{8,10}} & [0, 3, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 4]_{\varpi} \\
 &+ q^{10} \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\gamma \alpha_{8,10}} & [0, 3, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 4]_{\varpi}
 \end{aligned}$$

Reverse decomposition

$$\begin{aligned}
 \mathbf{M}_{\lambda}^{\gamma \alpha_{8,10}} &= \mathbf{M}_{\lambda}^{\gamma \alpha_{8,10}} & [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\
 &- q^7 \mathbf{M}_{\lambda - \alpha_{5,10} - \alpha_{4,9} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\gamma \alpha_{8,10}} & [0, 3, 2, 2, 0, 0, 0, 0, 0, 0, 1, 0, 1, 4]_{\varpi} \\
 &- q^4 \mathbf{M}_{\lambda - \alpha_{5,14}}^{\gamma \alpha_{8,10}} & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{7,9}}^{\gamma \alpha_{8,10}} & [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, 0, 1, 3]_{\varpi} \\
 &- q^7 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13}}^{\gamma \alpha_{8,10}} & [0, 3, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 4]_{\varpi} \\
 &+ q^{10} \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\gamma \alpha_{8,10}} & [0, 3, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 4]_{\varpi}
 \end{aligned}$$

Positive decomposition

$$\begin{aligned}
 \mathbf{M}_{\lambda}^{\succ \alpha_{8,10}} &= \mathbf{M}_{\lambda}^{\succ \alpha_{8,10}} & [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\
 &+ q^7 \mathbf{M}_{\lambda - \alpha_{5,10} - \alpha_{4,9} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} & [0, 3, 2, 2, 0, 0, 0, 0, 0, 0, 1, 0, 1, 4]_{\varpi} \\
 &+ q^4 \mathbf{M}_{\lambda - \alpha_{5,14}}^{\succ \alpha_{8,10}} & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &- q^5 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{7,9}}^{\succ \alpha_{8,10}} & [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, 0, 1, 3]_{\varpi} \\
 &+ q^7 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13}}^{\succ \alpha_{8,10}} & [0, 3, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 4]_{\varpi} \\
 &- q^{10} \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} & [0, 3, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 4]_{\varpi}
 \end{aligned}$$

Positive decomposition

$$\begin{aligned}
 \mathbf{M}_{\lambda}^{\succ \alpha_{8,10}} &= \mathbf{M}_{\lambda}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\
 &+ q^7 \mathbf{M}_{\lambda - \alpha_{5,10} - \alpha_{4,9} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} & [0, 3, 2, 2, 0, 0, 0, 0, 0, 1, 0, 1, 4]_{\varpi} \\
 &+ q^4 \mathbf{M}_{\lambda - \alpha_{5,14}}^{\succ \alpha_{8,10}} & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &- q^5 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{7,9}}^{\succ \alpha_{8,10}} & [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, 0, 1, 3]_{\varpi} \\
 &+ q^7 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13}}^{\succ \alpha_{8,10}} & [0, 3, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 4]_{\varpi} \\
 &- q^{10} \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} & [0, 3, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 4]_{\varpi}
 \end{aligned}$$

Positive decomposition

$$\begin{aligned}
 \mathbf{M}_{\lambda}^{\succ \alpha_{8,10}} &= \mathbf{M}_{\lambda}^{\succ \alpha_{8,10}} & [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\
 &+ q^7 \mathbf{M}_{\lambda - \alpha_{5,10} - \alpha_{4,9} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} & [0, 3, 2, 2, 0, 0, 0, 0, 0, 0, 1, 0, 1, 4]_{\varpi} \\
 &+ q^4 \mathbf{M}_{\lambda - \alpha_{5,14}}^{\succ \alpha_{8,10}} & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &- q^5 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{7,9}}^{\succ \alpha_{8,10}} & [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, 0, 1, 3]_{\varpi} \\
 &+ q^7 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13}}^{\succ \alpha_{8,10}} & [0, 3, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 4]_{\varpi} \\
 &- q^{10} \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} & [0, 3, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 4]_{\varpi}
 \end{aligned}$$

Positive decomposition for good pairs

$$\mathbf{M}_{\lambda-\alpha_{5,14}}^{\gamma_{\alpha_{8,10}}} - q\mathbf{M}_{\lambda-\alpha_{5,14}-\alpha_{7,9}}^{\gamma_{\alpha_{8,10}}} = \mathbf{M}_{\lambda-\alpha_{5,14}}^{\gamma_{\alpha_{7,9}}} [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 1, 3]_{\varpi}$$

Positive decomposition for good pairs

$$\begin{aligned} \mathbf{M}_{\lambda-\alpha_{5,14}}^{\succ \alpha_{8,10}} - q \mathbf{M}_{\lambda-\alpha_{5,14}-\alpha_{7,9}}^{\succ \alpha_{8,10}} &= \mathbf{M}_{\lambda-\alpha_{5,14}}^{\succ \alpha_{7,9}} & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\ &+ q^2 \mathbf{M}_{\lambda-\alpha_{5,14}-\alpha_{8,10}}^{\succ \alpha_{7,9}} & [0, 3, 0, 3, 0, 2, 1, -1, 0, -1, 1, 0, 1, 3]_{\varpi} \end{aligned}$$

Positive decomposition for good pairs

$$\begin{aligned} \mathbf{M}_{\lambda-\alpha_{5,14}}^{\succ \alpha_{8,10}} - q \mathbf{M}_{\lambda-\alpha_{5,14}-\alpha_{7,9}}^{\succ \alpha_{8,10}} &= \mathbf{M}_{\lambda-\alpha_{5,14}}^{\succ \alpha_{7,9}} && [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\ &+ q^5 \mathbf{M}_{\lambda-\alpha_{5,14}-\alpha_{2,14}}^{\succ \alpha_{7,9}} && [1, 2, 0, 3, 0, 2, 0, 0, 0, 0, 0, 1, 2]_{\varpi} \end{aligned}$$

Positive decomposition for good pairs

$$\begin{aligned}
 \mathbf{M}_{\lambda-\alpha_{5,14}}^{\succ \alpha_{8,10}} - q \mathbf{M}_{\lambda-\alpha_{5,14}-\alpha_{7,9}}^{\succ \alpha_{8,10}} &= \mathbf{M}_{\lambda-\alpha_{5,14}}^{\succ \alpha_{7,9}} & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 3] \varpi \\
 &+ q^5 \mathbf{M}_{\lambda-\alpha_{5,14}-\alpha_{2,14}}^{\succ \alpha_{7,9}} & [1, 2, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 2] \varpi \\
 &+ q^{10} \mathbf{M}_{\lambda-\alpha_{5,14}-2\alpha_{2,14}}^{\succ \alpha_{7,9}} & [2, 1, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 1] \varpi
 \end{aligned}$$

Positive decomposition for good pairs

$$\begin{aligned}
 \mathbf{M}_{\lambda-\alpha_{5,14}}^{\succ\alpha_{8,10}} - q\mathbf{M}_{\lambda-\alpha_{5,14}-\alpha_{7,9}}^{\succ\alpha_{8,10}} &= \mathbf{M}_{\lambda-\alpha_{5,14}}^{\succ\alpha_{7,9}} & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\lambda-\alpha_{5,14}-\alpha_{2,14}}^{\succ\alpha_{7,9}} & [1, 2, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 2]_{\varpi} \\
 &+ q^{10} \mathbf{M}_{\lambda-\alpha_{5,14}-2\alpha_{2,14}}^{\succ\alpha_{7,9}} & [2, 1, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 1]_{\varpi} \\
 &+ q^{15} \mathbf{M}_{\lambda-\alpha_{5,14}-3\alpha_{2,14}}^{\succ\alpha_{7,9}} & [3, 0, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 0]_{\varpi}
 \end{aligned}$$

Positive decomposition for good pairs

$$\begin{aligned}
 \mathbf{M}_{\lambda-\alpha_{5,14}}^{\succ\alpha_{8,10}} - q\mathbf{M}_{\lambda-\alpha_{5,14}-\alpha_{7,9}}^{\succ\alpha_{8,10}} &= \mathbf{M}_{\lambda-\alpha_{5,14}}^{\succ\alpha_{7,9}} & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\lambda-\alpha_{5,14}-\alpha_{2,14}}^{\succ\alpha_{7,9}} & [1, 2, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 2]_{\varpi} \\
 &+ q^{10} \mathbf{M}_{\lambda-\alpha_{5,14}-2\alpha_{2,14}}^{\succ\alpha_{7,9}} & [2, 1, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 1]_{\varpi} \\
 &+ q^{15} \mathbf{M}_{\lambda-\alpha_{5,14}-3\alpha_{2,14}}^{\succ\alpha_{7,9}} & [3, 0, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 0]_{\varpi} \\
 &+ q^{20} \mathbf{M}_{\lambda-\alpha_{5,14}-4\alpha_{2,14}}^{\succ\alpha_{7,9}} & [4, -1, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, -1]_{\varpi}
 \end{aligned}$$

Positive decomposition for good pairs

$$\begin{aligned}
 \mathbf{M}_{\lambda-\alpha_{5,14}}^{\succ\alpha_{8,10}} - q\mathbf{M}_{\lambda-\alpha_{5,14}-\alpha_{7,9}}^{\succ\alpha_{8,10}} &= \mathbf{M}_{\lambda-\alpha_{5,14}}^{\succ\alpha_{7,9}} & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\lambda-\alpha_{5,14}-\alpha_{2,14}}^{\succ\alpha_{7,9}} & [1, 2, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 2]_{\varpi} \\
 &+ q^{10} \mathbf{M}_{\lambda-\alpha_{5,14}-2\alpha_{2,14}}^{\succ\alpha_{7,9}} & [2, 1, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 1]_{\varpi} \\
 &+ q^{15} \mathbf{M}_{\lambda-\alpha_{5,14}-3\alpha_{2,14}}^{\succ\alpha_{7,9}} & [3, 0, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 0]_{\varpi}
 \end{aligned}$$

Positive decomposition

$$\begin{aligned}
 \mathbf{M}_{\lambda}^{\succ \alpha_{8,10}} &= \mathbf{M}_{\lambda}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\
 &+ q^7 \mathbf{M}_{\lambda - \alpha_{5,10} - \alpha_{4,9} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} & [0, 3, 2, 2, 0, 0, 0, 0, 0, 1, 0, 1, 4]_{\varpi} \\
 &+ q^4 \mathbf{M}_{\lambda - \alpha_{5,14}}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &- q^5 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{7,9}}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, 0, 1, 3]_{\varpi} \\
 &+ q^7 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13}}^{\succ \alpha_{8,10}} & [0, 3, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 4]_{\varpi} \\
 &- q^{10} \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} & [0, 3, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 4]_{\varpi}
 \end{aligned}$$

Positive decomposition

$$\begin{aligned}
 \mathbf{M}_{\lambda}^{\succ \alpha_{8,10}} &= \mathbf{M}_{\lambda}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\
 &+ q^7 \mathbf{M}_{\lambda - \alpha_{5,10} - \alpha_{4,9} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} & [0, 3, 2, 2, 0, 0, 0, 0, 0, 1, 0, 1, 4]_{\varpi} \\
 &+ q^4 \mathbf{M}_{\lambda - \alpha_{5,14}}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
 &- q^5 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{7,9}}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, 0, 1, 3]_{\varpi} \\
 &+ q^7 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13}}^{\succ \alpha_{8,10}} & [0, 3, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 4]_{\varpi} \\
 &- q^{10} \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} & [0, 3, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 4]_{\varpi}
 \end{aligned}$$

Positivity for bad pairs

Since both $\lambda - \alpha_{5,14} - \alpha_{4,13}$ and $\lambda - \alpha_{5,14} - \alpha_{4,13} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}$ are dominants, then we can get his positive decomposition!

$$M_{\lambda - \alpha_{5,14} - \alpha_{4,13}}^{\succ \alpha_{8,10}} [0, 3, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 4] \varpi$$

$$M_{\lambda - \alpha_{5,14} - \alpha_{4,13} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} [0, 3, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 4] \varpi$$

Positivity for bad pairs

Set $\mu = \lambda - \alpha_{5,14} - \alpha_{4,13}$ and $\nu = \lambda - \alpha_{5,14} - \alpha_{4,13} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}$.

$$\begin{aligned}
 \mathbf{M}_{\mu}^{\succ \alpha_{8,10}} &= \mathbf{M}_{\mu}^{\succ \alpha_{8,10}} & [0, 3, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 4]_{\varpi} \\
 &+ q^8 \mathbf{M}_{\mu - \alpha_{2,10} - \alpha_{4,9} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} & [1, 2, 3, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 4]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\mu - \alpha_{2,14}}^{\succ \alpha_{8,10}} & [1, 2, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 3]_{\varpi} \\
 &- q^6 \mathbf{M}_{\mu - \alpha_{2,14} - \alpha_{7,9}}^{\succ \alpha_{8,10}} & [1, 2, 1, 2, 0, 3, -1, 0, -1, 1, 0, 0, 0, 3]_{\varpi} \\
 \\
 \mathbf{M}_{\nu}^{\succ \alpha_{8,10}} &= \mathbf{M}_{\nu}^{\succ \alpha_{8,10}} & [0, 3, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 4]_{\varpi} \\
 &+ q^7 \mathbf{M}_{\nu - \alpha_{2,10} - \alpha_{4,9}}^{\succ \alpha_{8,10}} & [1, 2, 3, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 4]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\nu - \alpha_{2,14}}^{\succ \alpha_{8,10}} & [1, 2, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 3]_{\varpi} \\
 &- q^6 \mathbf{M}_{\nu - \alpha_{2,14} - \alpha_{7,9}}^{\succ \alpha_{8,10}} & [1, 2, 2, 2, 0, 1, -1, 0, 0, 1, 0, 0, 0, 3]_{\varpi}
 \end{aligned}$$

Positivity for bad pairs

Set $\mu = \lambda - \alpha_{5,14} - \alpha_{4,13}$ and $\nu = \lambda - \alpha_{5,14} - \alpha_{4,13} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}$.

$$\begin{aligned}
 \mathbf{M}_{\mu}^{\succ \alpha_{8,10}} &= \mathbf{M}_{\mu}^{\succ \alpha_{8,10}} && [0, 3, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 4]_{\varpi} \\
 &+ q^8 \mathbf{M}_{\mu - \alpha_{2,10} - \alpha_{4,9} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} \checkmark && [1, 2, 3, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 4]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\mu - \alpha_{2,14}}^{\succ \alpha_{8,10}} && [1, 2, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 3]_{\varpi} \\
 &- q^6 \mathbf{M}_{\mu - \alpha_{2,14} - \alpha_{7,9}}^{\succ \alpha_{8,10}} && [1, 2, 1, 2, 0, 3, -1, 0, -1, 1, 0, 0, 0, 3]_{\varpi} \\
 \\
 \mathbf{M}_{\nu}^{\succ \alpha_{8,10}} &= \mathbf{M}_{\nu}^{\succ \alpha_{8,10}} && [0, 3, 2, 2, 0, 0, 0, 0, 0, 1, 0, 0, 0, 4]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\nu - \alpha_{2,10} - \alpha_{4,9}}^{\succ \alpha_{8,10}} \checkmark && [1, 2, 3, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 4]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\nu - \alpha_{2,14}}^{\succ \alpha_{8,10}} && [1, 2, 2, 2, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 3]_{\varpi} \\
 &- q^6 \mathbf{M}_{\nu - \alpha_{2,14} - \alpha_{7,9}}^{\succ \alpha_{8,10}} && [1, 2, 2, 2, 0, 1, -1, 0, 0, 1, 0, 0, 0, 0, 3]_{\varpi}
 \end{aligned}$$

Positivity for bad pairs

$$\begin{aligned}
 \mathbf{M}_{\mu-\alpha_{2,14}}^{\succ\alpha_{8,10}} - q\mathbf{M}_{\mu-\alpha_{2,14}-\alpha_{7,9}}^{\succ\alpha_{8,10}} &= \mathbf{M}_{\mu-\alpha_{2,14}}^{\succ\alpha_{7,9}} & [1, 2, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 3]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\mu-2\alpha_{2,14}}^{\succ\alpha_{7,9}} & [2, 1, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 2]_{\varpi} \\
 &+ q^{10} \mathbf{M}_{\mu-3\alpha_{2,14}}^{\succ\alpha_{7,9}} & [3, 0, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 1]_{\varpi}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{M}_{\nu-\alpha_{2,14}}^{\succ\alpha_{8,10}} - q\mathbf{M}_{\nu-\alpha_{2,14}-\alpha_{7,9}}^{\succ\alpha_{8,10}} &= \mathbf{M}_{\nu-\alpha_{2,14}}^{\succ\alpha_{7,9}} & [1, 2, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 3]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\nu-2\alpha_{2,14}}^{\succ\alpha_{7,9}} & [2, 1, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 2]_{\varpi} \\
 &+ q^{10} \mathbf{M}_{\nu-3\alpha_{2,14}}^{\succ\alpha_{7,9}} & [3, 0, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1]_{\varpi}
 \end{aligned}$$

Positivity for bad pairs

$$\begin{aligned}
 \mathbf{M}_{\mu-2\alpha_{2,14}}^{\succ\alpha_{7,9}} &= \mathbf{M}_{\mu-2\alpha_{2,14}}^{\succ\alpha_{6,8}} & [2, 1, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 2]_{\varpi} \\
 &+ q^3 \mathbf{M}_{\mu-2\alpha_{2,14}-\alpha_{6,8}-\alpha_{5,7}-\alpha_{4,6}}^{\succ\alpha_{7,9}} & [2, 1, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 2]_{\varpi} \\
 &+ q^4 \mathbf{M}_{\mu-2\alpha_{2,14}-\alpha_{6,14}}^{\succ\alpha_{7,9}} \checkmark & [2, 1, 1, 2, 1, 1, 0, 0, 0, 0, 0, 0, 1]_{\varpi} \\
 &- q^5 \mathbf{M}_{\mu-2\alpha_{2,14}-\alpha_{6,14}-\alpha_{5,7}}^{\succ\alpha_{7,9}} \checkmark & [2, 1, 1, 3, 0, 1, -1, 1, 0, 0, 0, 0, 1]_{\varpi} \\
 \\
 &\mathbf{M}_{\nu-2\alpha_{2,14}}^{\succ\alpha_{7,9}} & [2, 1, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 2]_{\varpi}
 \end{aligned}$$

Positivity for bad pairs

$$\begin{aligned}
\mathbf{M}_{\mu-3\alpha_{2,14}}^{\succeq\alpha_{7,9}} &= \mathbf{M}_{\mu-3\alpha_{2,14}}^{\succeq\alpha_{6,8}} & [3, 0, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 1]_{\varpi} \\
&+ q^3 \mathbf{M}_{\mu-3\alpha_{2,14}-\alpha_{6,8}-\alpha_{5,7}-\alpha_{4,6}}^{\succeq\alpha_{7,9}} & [3, 0, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1]_{\varpi} \\
&+ q^4 \mathbf{M}_{\mu-3\alpha_{2,14}-\alpha_{6,14}}^{\succeq\alpha_{7,9}} & [3, 0, 1, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0]_{\varpi} \\
&- q^5 \mathbf{M}_{\mu-3\alpha_{2,14}-\alpha_{6,14}-\alpha_{5,7}}^{\succeq\alpha_{7,9}} & [3, 0, 1, 3, 0, 1, -1, 1, 0, 0, 0, 0, 0, 0]_{\varpi} \\
\\
\mathbf{M}_{\nu-3\alpha_{2,14}}^{\succeq\alpha_{7,9}} & & [3, 0, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1]_{\varpi}
\end{aligned}$$

Positivity for bad pairs

$$\begin{aligned}
 \mathbf{M}_{\mu-3\alpha_{2,14}}^{\succ\alpha_{7,9}} &= \mathbf{M}_{\mu-3\alpha_{2,14}}^{\succ\alpha_{6,8}} & [3, 0, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 1]_{\varpi} \\
 &+ q^3 \mathbf{M}_{\mu-3\alpha_{2,14}-\alpha_{6,8}-\alpha_{5,7}-\alpha_{4,6}}^{\succ\alpha_{7,9}} & [3, 0, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1]_{\varpi} \\
 &+ q^4 \mathbf{M}_{\mu-3\alpha_{2,14}-\alpha_{6,14}}^{\succ\alpha_{7,9}} \checkmark & [3, 0, 1, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0]_{\varpi} \\
 &- q^5 \mathbf{M}_{\mu-3\alpha_{2,14}-\alpha_{6,14}-\alpha_{5,7}}^{\succ\alpha_{7,9}} \checkmark & [3, 0, 1, 3, 0, 1, -1, 1, 0, 0, 0, 0, 0, 0]_{\varpi} \\
 \\
 &\mathbf{M}_{\nu-3\alpha_{2,14}}^{\succ\alpha_{7,9}} & [3, 0, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1]_{\varpi}
 \end{aligned}$$

Positivity for bad pairs

$$\begin{aligned}
 \mathbf{M}_{\mu-\alpha_{2,14}}^{\succ\alpha_{8,10}} - q\mathbf{M}_{\mu-\alpha_{2,14}-\alpha_{7,9}}^{\succ\alpha_{8,10}} &= \mathbf{M}_{\mu-\alpha_{2,14}}^{\succ\alpha_{7,9}} && [1, 2, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 3]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\mu-2\alpha_{2,14}}^{\succ\alpha_{7,9}} \checkmark && [2, 1, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 2]_{\varpi} \\
 &+ q^{10} \mathbf{M}_{\mu-3\alpha_{2,14}}^{\succ\alpha_{7,9}} \checkmark && [3, 0, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 1]_{\varpi} \\
 \\
 \mathbf{M}_{\nu-\alpha_{2,14}}^{\succ\alpha_{8,10}} - q\mathbf{M}_{\nu-\alpha_{2,14}-\alpha_{7,9}}^{\succ\alpha_{8,10}} &= \mathbf{M}_{\nu-\alpha_{2,14}}^{\succ\alpha_{7,9}} && [1, 2, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 3]_{\varpi} \\
 &+ q^5 \mathbf{M}_{\nu-2\alpha_{2,14}}^{\succ\alpha_{7,9}} \checkmark && [2, 1, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 2]_{\varpi} \\
 &+ q^{10} \mathbf{M}_{\nu-3\alpha_{2,14}}^{\succ\alpha_{7,9}} \checkmark && [3, 0, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1]_{\varpi}
 \end{aligned}$$

Positivity for bad pairs

$$\begin{aligned}
 \mathbf{M}_{\mu-\alpha_{2,14}}^{\succeq\alpha_{7,9}} &= \mathbf{M}_{\mu-\alpha_{2,14}}^{\succeq\alpha_{6,8}} & [1, 2, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 3]_{\varpi} \\
 &+ q^3 \mathbf{M}_{\mu-\alpha_{2,14}-\alpha_{6,8}-\alpha_{5,7}-\alpha_{4,6}}^{\succeq\alpha_{7,9}} & [1, 2, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 3]_{\varpi} \\
 &+ q^4 \mathbf{M}_{\mu-\alpha_{2,14}-\alpha_{6,14}}^{\succeq\alpha_{6,8}} & [1, 2, 1, 2, 1, 1, 0, 0, 0, 0, 0, 0, 2]_{\varpi} \\
 &- q^5 \mathbf{M}_{\mu-\alpha_{2,14}-\alpha_{6,14}-\alpha_{5,7}}^{\succeq\alpha_{6,8}} & [1, 2, 1, 3, 0, 1, -1, 1, 0, 0, 0, 0, 2]_{\varpi} \\
 &+ \mathbf{M}_{\nu-\alpha_{2,14}}^{\succeq\alpha_{7,9}} & [1, 2, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 3]_{\varpi}
 \end{aligned}$$

Positivity for bad pairs

$$\begin{aligned}
 \mathbf{M}_{\mu-\alpha_{2,14}}^{\succeq \alpha_{7,9}} &= \mathbf{M}_{\mu-\alpha_{2,14}}^{\succeq \alpha_{6,8}} & [1, 2, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 3]_{\varpi} \\
 &+ q^3 \mathbf{M}_{\mu-\alpha_{2,14}-\alpha_{6,8}-\alpha_{5,7}-\alpha_{4,6}}^{\succeq \alpha_{7,9}} \checkmark & [1, 2, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 3]_{\varpi} \\
 &+ q^4 \mathbf{M}_{\mu-\alpha_{2,14}-\alpha_{6,14}}^{\succeq \alpha_{6,8}} \checkmark & [1, 2, 1, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 2]_{\varpi} \\
 &- q^5 \mathbf{M}_{\mu-\alpha_{2,14}-\alpha_{6,14}-\alpha_{5,7}}^{\succeq \alpha_{6,8}} \checkmark & [1, 2, 1, 3, 0, 1, -1, 1, 0, 0, 0, 0, 0, 2]_{\varpi} \\
 \\
 &\mathbf{M}_{\nu-\alpha_{2,14}}^{\succeq \alpha_{7,9}} \checkmark & [1, 2, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 3]_{\varpi}
 \end{aligned}$$

Positive decomposition

$$\begin{aligned}
 \mathbf{M}_{\lambda}^{\succ \alpha_{8,10}} &= \mathbf{M}_{\lambda}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 0, 1, 4] \varpi \\
 &+ q^7 \mathbf{M}_{\lambda - \alpha_{5,10} - \alpha_{4,9} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} & [0, 3, 2, 2, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 4] \varpi \\
 &+ q^4 \mathbf{M}_{\lambda - \alpha_{5,14}}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 3] \varpi \\
 &- q^5 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{7,9}}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, 0, 1, 3] \varpi \\
 &+ q^7 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13}}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 4] \varpi \\
 &- q^{10} \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 4] \varpi
 \end{aligned}$$

Positive decomposition

$$\begin{aligned}
\mathbf{M}_\lambda^{\succ \alpha_{8,10}} &= \mathbf{M}_\lambda^{\succ \alpha_{8,10}} \checkmark & [0, 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 0, 1, 4]_{\varpi} \\
&+ q^7 \mathbf{M}_{\lambda - \alpha_{5,10} - \alpha_{4,9} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 2, 2, 0, 0, 0, 0, 0, 0, 1, 0, 1, 4]_{\varpi} \\
&+ q^4 \mathbf{M}_{\lambda - \alpha_{5,14}}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 0, 3, 0, 2, 0, 0, 0, 0, 0, 0, 1, 3]_{\varpi} \\
&- q^5 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{7,9}}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 0, 3, 0, 3, -1, 0, -1, 1, 0, 0, 1, 3]_{\varpi} \\
&+ q^7 \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13}}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 1, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 4]_{\varpi} \\
&- q^{10} \mathbf{M}_{\lambda - \alpha_{5,14} - \alpha_{4,13} - \alpha_{6,8} - \alpha_{5,7} - \alpha_{4,6}}^{\succ \alpha_{8,10}} \checkmark & [0, 3, 2, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 4]_{\varpi}
\end{aligned}$$

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Thank you for your attention